1 Introduction

1.1 Angular Spectrum of Plane Waves

A Gaussian beam with a beam waist of w_0 has a corresponding angular representation

$$A_0(\theta, \phi) = \frac{1}{N}\, e^{-\left(\frac{\theta}{\theta_0}\right)^2} \tag{1}$$

where

$$\theta_0 = \frac{\lambda}{\pi w_0} \tag{2}$$

and the normalization N is chosen so that

$$\int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} A_0(\theta, \phi)^2 \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi = 1 \tag{3}$$

so that the coupling integral, I_{00}, for any beam with itself is unity

$$I_{ij} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi/2} A_i^*(\theta, \phi) A_j(\theta, \phi) \sin\theta \,\mathrm{d}\theta \,\mathrm{d}\phi \tag{4}$$

1.2 Propagation of the Beam

Each infinite plane wave, propagating through a distance z in a vacuum will undergo a phase shift $\vec{k} \cdot \vec{z}$. The following equations were implemented in the Octave function `propagatebeam.m`.

$$A'(\theta, \phi) = A(\theta, \phi)\, e^{ikz\cos\theta} \tag{5}$$

$$k = \frac{2\pi\nu}{c} \tag{6}$$

For propagation through layers of material the Octave function `layers_complex.m` was used. This returns amplitude transmission coefficients t_{TE} and t_{TM} for a stack of layers with given complex values of ε and μ. The TE and TM components of each plane wave were identified as $A(\theta, \phi)\cos\phi$ and $A(\theta, \phi)\sin\phi$ respectively. Then using the same relations to re-assemble the beam, propagation through the layers is given by:

$$\begin{aligned} A'(\theta, \phi) &= \cos^2(\phi)\, t_{\mathrm{TE}}(\theta, z, \nu)\, A(\theta, \phi) \\ &+ \sin^2(\phi)\, t_{\mathrm{TM}}(\theta, z, \nu)\, A(\theta, \phi) \end{aligned} \tag{7}$$

The Octave function `layersbeam.m` performs this calculation.

2 Test Runs

A number of test runs were performed as a function of layer thickness, z. In each case a set of beam coefficients, A_0, for a given angle, θ_0, was generated. The beam was the propagated both using the `layers_complex.m` code and also the simple beam propagation function for the same distance.

$$A_1 = \texttt{layersbeam.m} \quad \nu, \varepsilon, z, A_0 \tag{8}$$

$$A_2 = \texttt{propagatebeam.m} \quad \nu, z, A_0 \tag{9}$$

Finally the magnitude of the coupling integral was taken

$$|I_{12}| \tag{10}$$

which should be equal to 1 for an plane wave in air due to the correction for distance applied to A_2.

To test the routines, this procedure was carried out for vacuum ($\varepsilon = 1$), giving the result shown in Figure 1. As expected this gives a value of $I_{12} = 1$ for each distance and beam angle.

The procedure was also tested for values of $\varepsilon = 2, \quad 4$ ($n = \sqrt{2}, \quad 2$) giving the results shown in Figure 2 and Figure 3.

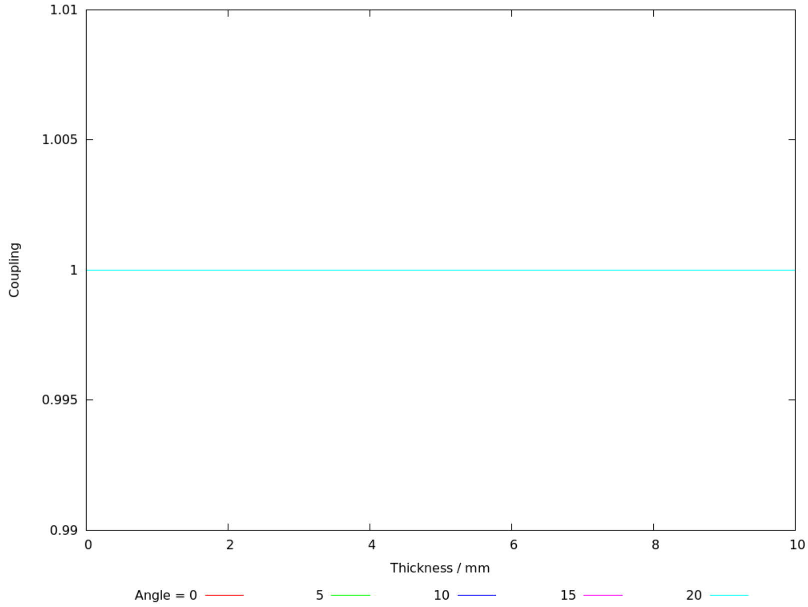

Figure 1: $\varepsilon = 1$

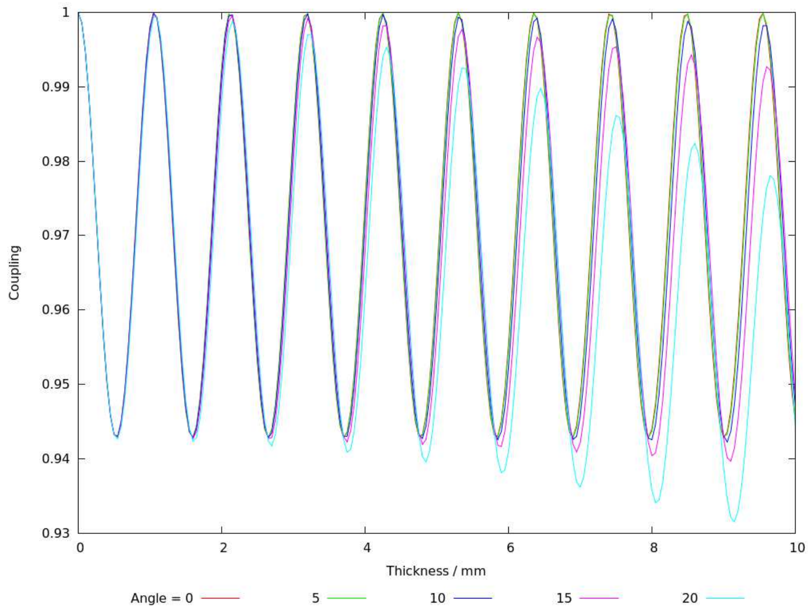

Figure 2: $\varepsilon = 2$

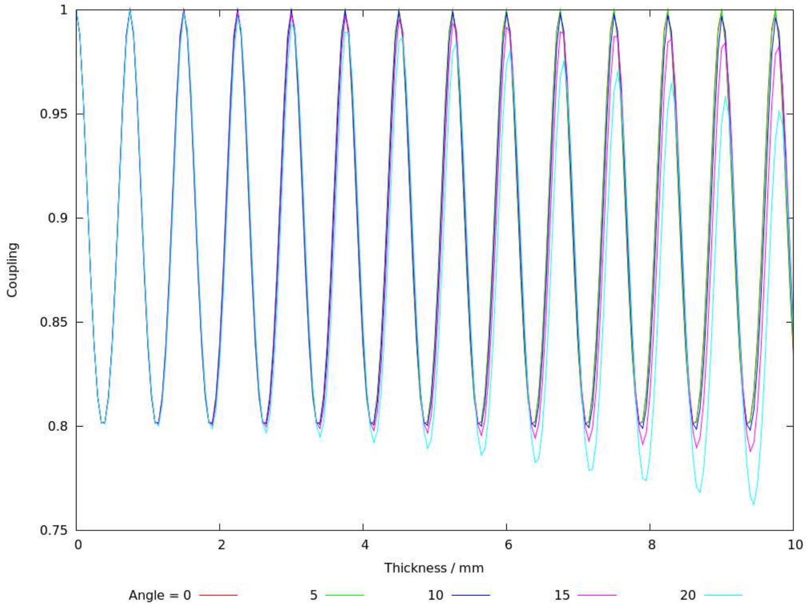

Figure 3: $\varepsilon = 4$

3 Results

The routine was then run as a function of frequency with the following parameters:

- Refractive index $n = 2$ ($\varepsilon = 4$)
- Thickness $z = 10$ mm
- Frequency $\nu = 75 - 110$ GHz

The results are plotted in Figure 4. Finally the period of the oscillations was fitted in order to determine the refractive index of the material, giving the plot shown in Figure 5.

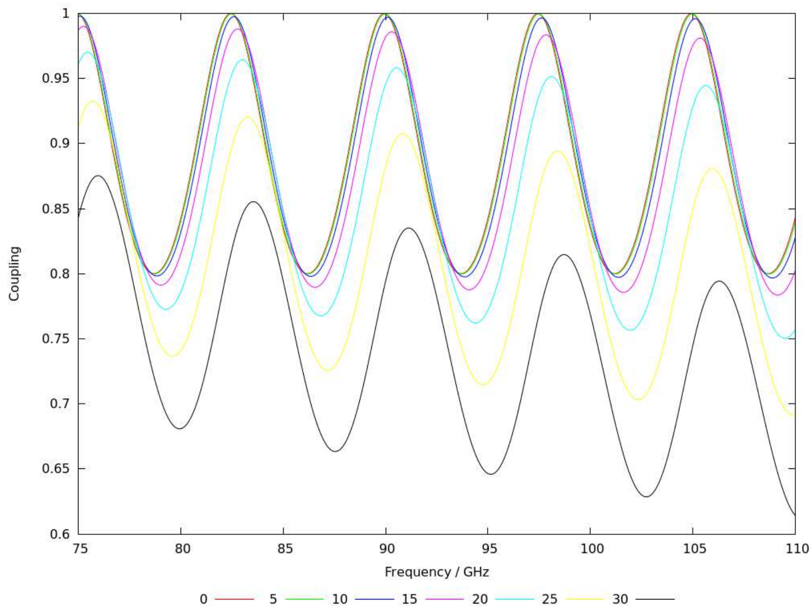

Figure 4: $\varepsilon = 4$

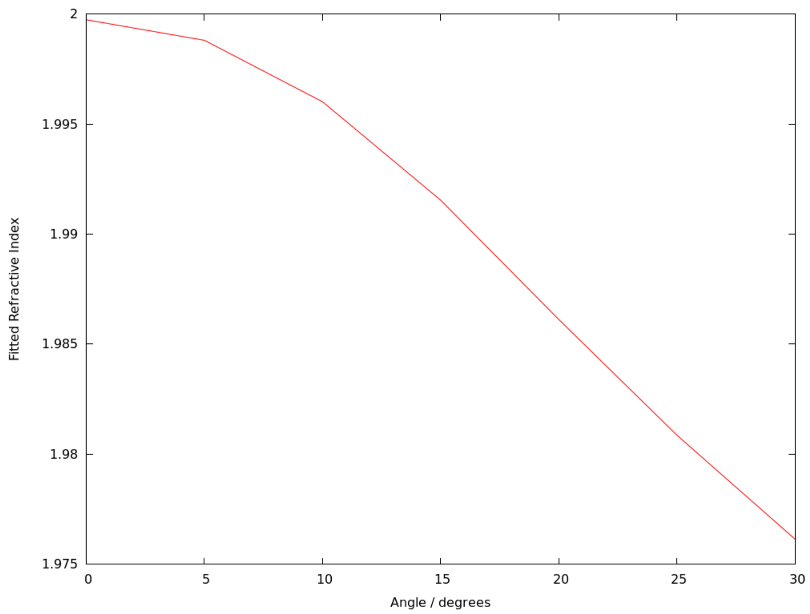

Figure 5: Fitted refractive index for each angle θ_0.

A Numerical Evaluation of Coupling Integral

A.1 Evaluation as a Sum

Sample points are defined as follows, where in order to handle the case of a pure plane wave $\theta = 0$ represents the range $0 \leq \theta < \Delta\theta/2$ and other values represent the range $\theta_i - \Delta\theta/2 \leq \theta < \theta_i + \Delta\theta/2$.

$$\theta: \quad 0 \quad \ldots \quad (n_\theta - 1)\tfrac{\pi}{2n_\theta - 1}, \quad \Delta\theta = \frac{\pi}{2n_\theta - 1} \tag{11}$$

$$\phi: \quad -\pi(1 - \tfrac{2}{n_\phi}) \quad \ldots \quad \pi, \quad \Delta\phi = \frac{2\pi}{n_\phi} \tag{12}$$

Then the surface area of the corresponding element unit sphere is given by

$$\begin{aligned}
\int_{\phi-\Delta\phi/2}^{\phi+\Delta\phi/2} \int_{\theta-\Delta\theta/2}^{\theta+\Delta\theta/2} \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi &= \Big[\phi\Big[-\cos\theta\Big]_{\theta-\Delta\theta/2}^{\theta+\Delta\theta/2}\Big]_{\phi-\Delta\phi/2}^{\phi+\Delta\phi/2} \\
&= \Delta\phi\left(\cos(\theta - \Delta\theta/2) - \cos(\theta + \Delta\theta/2)\right) \\
&= 2\sin\theta\,\sin(\Delta\theta/2)\,\Delta\phi
\end{aligned} \tag{13}$$

or in the case of $\theta = 0$

$$\Big[\phi\Big[-\cos\theta\Big]_{0}^{\Delta\theta/2}\Big]_{\phi-\Delta\phi/2}^{\phi+\Delta\phi/2} = (1 - \cos(\Delta\theta/2))\,\Delta\phi$$

So the final expression for discrete sample points is:

$$I_{ij} = \sum_{\phi=0}^{2\pi} \sum_{\theta=0}^{\pi/2} A_i^*(\theta,\phi) A_j(\theta,\phi) \begin{cases} (1 - \cos(\Delta\theta/2))\,\Delta\phi & \theta = 0 \\ 2\sin\theta\,\sin(\Delta\theta/2)\,\Delta\phi & \text{otherwise} \end{cases} \tag{14}$$

A.2 Minimum Number of Sample Points

To ensure a smooth variation of phase between the sample points we can impose a limit

$$|kz\cos\theta_{i+1} - kz\cos\theta_i| \ll \frac{\pi}{8} \tag{15}$$

which can be rearranged to

$$\Delta\theta\sin\theta \ll \frac{c}{16\nu z} \tag{16}$$

or taking the worst case ($\sin\theta = 1$) and a general medium

$$\Delta\theta \ll \frac{c}{16\nu z\sqrt{\varepsilon\mu}} \tag{17}$$

For example with $n = 2$, $\nu = 100$ GHz, $z = 1$ mm

$$\Delta\theta \ll 5^\circ \quad \Rightarrow \quad \gg 18 \text{ points} \tag{18}$$